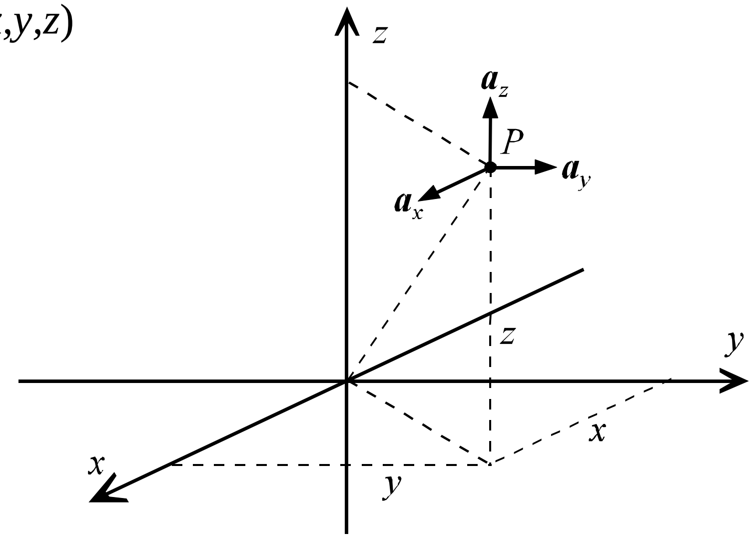
Coordinate and Unit Vector Definitions

Rectangular Coordinates (x,y,z)

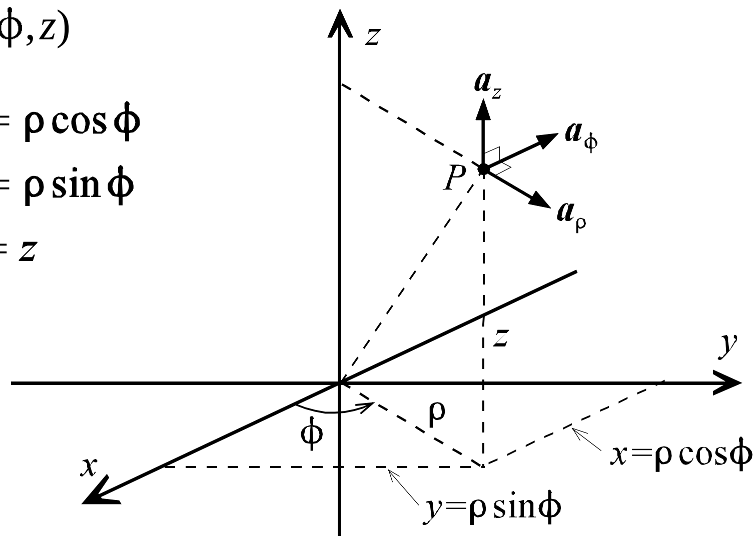
$$(-\infty < x < \infty)$$
$$(-\infty < y < \infty)$$
$$(-\infty < z < \infty)$$



Cylindrical Coordinates (ρ,ϕ,z)

$$\rho = \sqrt{x^2+y^2} \qquad x = \rho\cos\phi$$
$$\phi = \tan^{-1}(y/x) \qquad y = \rho\sin\phi$$
$$z = z \qquad z = z$$

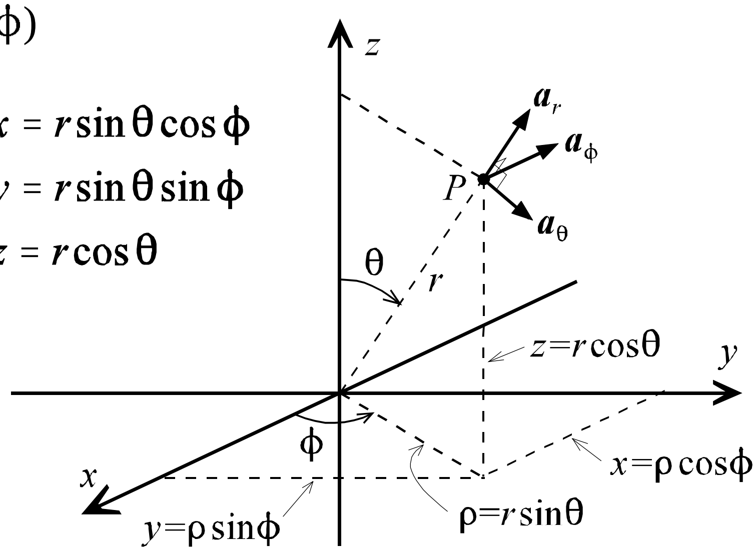
$$(0 \le \rho < \infty)$$
$$(0 \le \phi < 2\pi)$$
$$(-\infty < z < \infty)$$



Spherical Coordinates (r,θ,ϕ)

$$r = \sqrt{x^2+y^2+z^2} \qquad x = r\sin\theta\cos\phi$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$\phi = \tan^{-1}(y/x)$$

$$(0 \le r < \infty)$$
$$(0 \le \theta \le \pi)$$
$$(0 \le \phi < 2\pi)$$



Vector Definitions and Coordinate Transformations

Vector Definitions

Rectangular $\quad \boldsymbol{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z = (A_x, A_y, A_z)$

Cylindrical $\quad \boldsymbol{A} = A_\rho \boldsymbol{a}_\rho + A_\phi \boldsymbol{a}_\phi + A_z \boldsymbol{a}_z = (A_\rho, A_\phi, A_z)$

Spherical $\quad \boldsymbol{A} = A_r \boldsymbol{a}_r + A_\theta \boldsymbol{a}_\theta + A_\phi \boldsymbol{a}_\phi = (A_r, A_\theta, A_\phi)$

Vector Magnitudes

$$\boldsymbol{A}\cdot\boldsymbol{A} = |\boldsymbol{A}|\,|\boldsymbol{A}|\cos 0^o = |\boldsymbol{A}|^2 \qquad \Rightarrow \qquad |\boldsymbol{A}| = \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$$

Rectangular $\quad |\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Cylindrical $\quad |\boldsymbol{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

Spherical $\quad |\boldsymbol{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$

Rectangular to Cylindrical Coordinate Transformation

$$(A_x, A_y, A_z) \Rightarrow (A_\rho, A_\phi, A_z)$$

The transformation of rectangular to cylindrical coordinates requires that we find the components of the rectangular coordinate vector $\boldsymbol{A}$ in the direction of the cylindrical coordinate unit vectors (using the dot product). The required dot products are

$$A_\rho = \boldsymbol{A}\cdot\boldsymbol{a}_\rho = A_x\boldsymbol{a}_x\cdot\boldsymbol{a}_\rho + A_y\boldsymbol{a}_y\cdot\boldsymbol{a}_\rho + A_z\boldsymbol{a}_z\cdot\boldsymbol{a}_\rho = A_x\boldsymbol{a}_x\cdot\boldsymbol{a}_\rho + A_y\boldsymbol{a}_y\cdot\boldsymbol{a}_\rho$$
$$A_\phi = \boldsymbol{A}\cdot\boldsymbol{a}_\phi = A_x\boldsymbol{a}_x\cdot\boldsymbol{a}_\phi + A_y\boldsymbol{a}_y\cdot\boldsymbol{a}_\phi + A_z\boldsymbol{a}_z\cdot\boldsymbol{a}_\phi = A_x\boldsymbol{a}_x\cdot\boldsymbol{a}_\phi + A_y\boldsymbol{a}_y\cdot\boldsymbol{a}_\phi$$
$$A_z = \boldsymbol{A}\cdot\boldsymbol{a}_z = A_x\boldsymbol{a}_x\cdot\boldsymbol{a}_z + A_y\boldsymbol{a}_y\cdot\boldsymbol{a}_z + A_z\boldsymbol{a}_z\cdot\boldsymbol{a}_z = A_z$$

where the $\boldsymbol{a}_z$ unit vector is identical in both orthogonal coordinate systems

such that

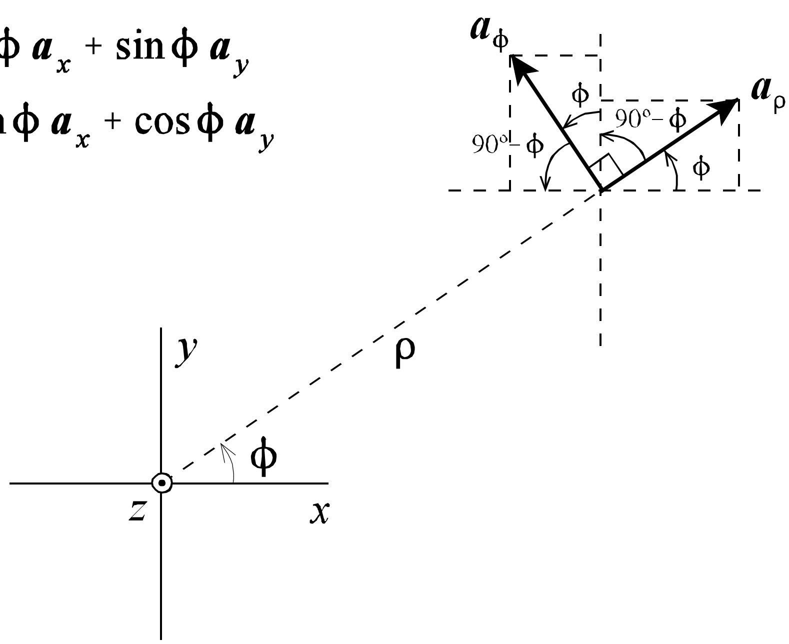
$$\boldsymbol{a}_z \cdot \boldsymbol{a}_\rho = 0 \qquad \boldsymbol{a}_z \cdot \boldsymbol{a}_\phi = 0$$

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_z = 0 \qquad \boldsymbol{a}_y \cdot \boldsymbol{a}_z = 0 \qquad \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$$

The four remaining unit vector dot products are determined according to the geometry relationships between the two coordinate systems.

$$\boldsymbol{a}_\rho = \cos\phi\, \boldsymbol{a}_x + \sin\phi\, \boldsymbol{a}_y$$

$$\boldsymbol{a}_\phi = -\sin\phi\, \boldsymbol{a}_x + \cos\phi\, \boldsymbol{a}_y$$



$$\boldsymbol{a}_x \cdot \boldsymbol{a}_\rho = \boldsymbol{a}_x \cdot (\cos\phi\, \boldsymbol{a}_x + \sin\phi\, \boldsymbol{a}_y) = \cos\phi$$

$$\boldsymbol{a}_y \cdot \boldsymbol{a}_\rho = \boldsymbol{a}_y \cdot (\cos\phi\, \boldsymbol{a}_x + \sin\phi\, \boldsymbol{a}_y) = \sin\phi$$

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_\phi = \boldsymbol{a}_x \cdot (-\sin\phi\, \boldsymbol{a}_x + \cos\phi\, \boldsymbol{a}_y) = -\sin\phi$$

$$\boldsymbol{a}_y \cdot \boldsymbol{a}_\phi = \boldsymbol{a}_y \cdot (-\sin\phi\, \boldsymbol{a}_x + \cos\phi\, \boldsymbol{a}_y) = \cos\phi$$

The resulting cylindrical coordinate vector is

$$\begin{aligned} \boldsymbol{A} &= A_\rho \boldsymbol{a}_\rho + A_\phi \boldsymbol{a}_\phi + A_z \boldsymbol{a}_z \\ &= (A_x \cos\phi + A_y \sin\phi)\boldsymbol{a}_\rho + (A_y \cos\phi - A_x \sin\phi)\boldsymbol{a}_\phi + A_z \boldsymbol{a}_z \end{aligned}$$

In matrix form, the rectangular to cylindrical transformation is

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

<u>Cylindrical to Rectangular Coordinate Transformation</u>

$$(A_\rho, A_\phi, A_z) \;\Rightarrow\; (A_x, A_y, A_z)$$

The transformation from cylindrical to rectangular coordinates can be determined as the inverse of the rectangular to cylindrical transformation.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

The cylindrical coordinate variables in the transformation matrix must be expressed in terms of rectangular coordinates.

$$\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

The resulting transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & -\dfrac{y}{\sqrt{x^2+y^2}} & 0 \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

The cylindrical to rectangular transformation can be written as

$$\begin{aligned} \boldsymbol{A} &= A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z \\ &= (A_\rho \cos\phi - A_\phi \sin\phi)\boldsymbol{a}_x + (A_\rho \sin\phi + A_\phi \cos\phi)\boldsymbol{a}_y + A_z \boldsymbol{a}_z \\ &= \left(A_\rho \frac{x}{\sqrt{x^2+y^2}} - A_\phi \frac{y}{\sqrt{x^2+y^2}} \right) \boldsymbol{a}_x \\ &\qquad + \left(A_\rho \frac{y}{\sqrt{x^2+y^2}} + A_\phi \frac{x}{\sqrt{x^2+y^2}} \right) \boldsymbol{a}_y \\ &\qquad\qquad + A_z \boldsymbol{a}_z \end{aligned}$$

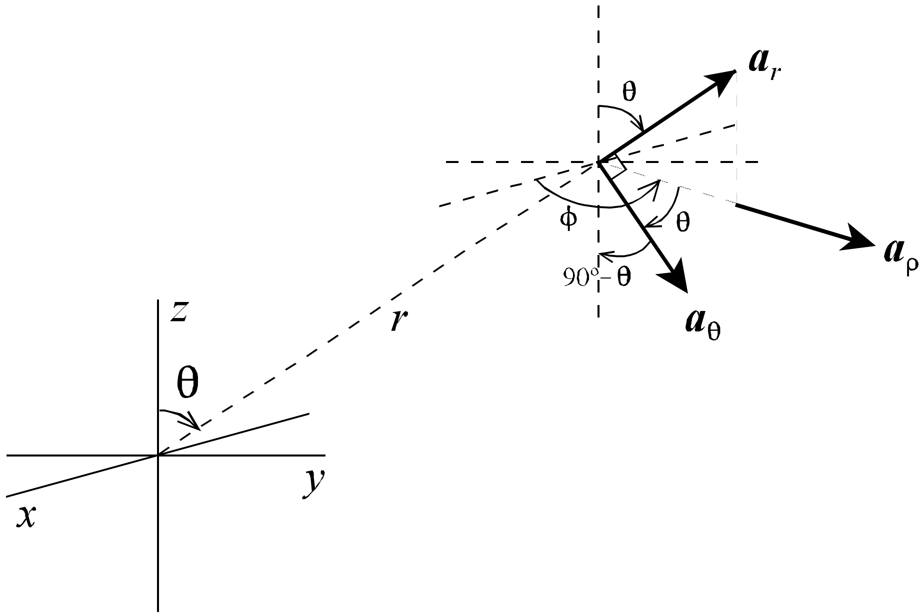
<u>Rectangular to Spherical Coordinate Transformation</u>

$$(A_x, A_y, A_z) \Rightarrow (A_r, A_\theta, A_\phi)$$

The dot products necessary to determine the transformation from rectangular coordinates to spherical coordinates are

$$\begin{aligned} A_r &= \boldsymbol{A} \cdot \boldsymbol{a}_r = A_x \boldsymbol{a}_x \cdot \boldsymbol{a}_r + A_y \boldsymbol{a}_y \cdot \boldsymbol{a}_r + A_z \boldsymbol{a}_z \cdot \boldsymbol{a}_r \\ A_\theta &= \boldsymbol{A} \cdot \boldsymbol{a}_\theta = A_x \boldsymbol{a}_x \cdot \boldsymbol{a}_\theta + A_y \boldsymbol{a}_y \cdot \boldsymbol{a}_\theta + A_z \boldsymbol{a}_z \cdot \boldsymbol{a}_\theta \\ A_\phi &= \boldsymbol{A} \cdot \boldsymbol{a}_\phi = A_x \boldsymbol{a}_x \cdot \boldsymbol{a}_\phi + A_y \boldsymbol{a}_y \cdot \boldsymbol{a}_\phi + A_z \boldsymbol{a}_z \cdot \boldsymbol{a}_\phi \end{aligned}$$

The geometry relationships between the rectangular and spherical unit vectors are illustrated below.



$$\boldsymbol{a}_r = \sin\theta\, \boldsymbol{a}_\rho + \cos\theta\, \boldsymbol{a}_z$$
$$= \sin\theta\, [\cos\phi\, \boldsymbol{a}_x + \sin\phi\, \boldsymbol{a}_y] + \cos\theta\, \boldsymbol{a}_z$$
$$= \sin\theta \cos\phi\, \boldsymbol{a}_x + \sin\theta \sin\phi\, \boldsymbol{a}_y + \cos\theta\, \boldsymbol{a}_z$$

$$\boldsymbol{a}_\theta = \cos\theta\, \boldsymbol{a}_\rho - \cos(90^o - \theta)\, \boldsymbol{a}_z$$
$$= \cos\theta\, [\cos\phi\, \boldsymbol{a}_x + \sin\phi\, \boldsymbol{a}_y] - \sin\theta\, \boldsymbol{a}_z$$
$$= \cos\theta \cos\phi\, \boldsymbol{a}_x + \cos\theta \sin\phi\, \boldsymbol{a}_y - \sin\theta\, \boldsymbol{a}_z$$

$$\boldsymbol{a}_\phi = -\sin\phi\, \boldsymbol{a}_x + \cos\phi\, \boldsymbol{a}_y$$

The dot products are then

$$\boldsymbol{a}_x \cdot \boldsymbol{a}_r = \sin\theta \cos\phi \qquad \boldsymbol{a}_y \cdot \boldsymbol{a}_r = \sin\theta \sin\phi \qquad \boldsymbol{a}_z \cdot \boldsymbol{a}_r = \cos\theta$$
$$\boldsymbol{a}_x \cdot \boldsymbol{a}_\theta = \cos\theta \cos\phi \qquad \boldsymbol{a}_y \cdot \boldsymbol{a}_\theta = \cos\theta \sin\phi \qquad \boldsymbol{a}_z \cdot \boldsymbol{a}_\theta = -\sin\theta$$
$$\boldsymbol{a}_x \cdot \boldsymbol{a}_\phi = -\sin\phi \qquad \boldsymbol{a}_y \cdot \boldsymbol{a}_\phi = \cos\phi \sin\phi \qquad \boldsymbol{a}_z \cdot \boldsymbol{a}_\phi = 0$$

and the rectangular to spherical transformation may be written as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

<u>Spherical to Rectangular Coordinate Transformation</u>

$(A_r, A_\theta, A_\phi) \Rightarrow (A_x, A_y, A_z)$

The spherical to rectangular coordinate transformation is the inverse of the rectangular to spherical coordinate transformation so that

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

The spherical coordinate variables in terms of the rectangular coordinate variables are

$$\sin\theta = \frac{\rho}{r} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \cos\theta = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$$

The complete spherical to rectangular coordinate transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} & -\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Dot Products of Unit Vectors

		Rectangular			Cylindrical			Spherical		
	$\bullet$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
Rectangular	$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
	$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
	$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\mathbf{a}_\rho$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
	$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\mathbf{a}_r$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

Coordinate and Coordinate Component Transformations

Rectangular to Cylindrical

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to Rectangular

$$\rho = \sqrt{x^2+y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Spherical to Rectangular

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & -\frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Rectangular to Spherical

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Cylindrical to Spherical

$$\rho = r\sin\theta$$
$$\phi = \phi$$
$$z = r\cos\theta$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Spherical to Cylindrical

$$r = \sqrt{\rho^2+z^2}$$
$$\theta = \tan^{-1}\frac{\rho}{z}$$
$$\phi = \phi$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{\rho}{\sqrt{\rho^2+z^2}} & \frac{z}{\sqrt{\rho^2+z^2}} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{\rho^2+z^2}} & -\frac{\rho}{\sqrt{\rho^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Coordinate Transformation Procedure

(1) Transform the component scalars into the new coordinate system.
(2) Insert the component scalars into the coordinate transformation matrix and evaluate.

Steps (1) and (2) can be performed in either order.

Example (Coordinate Transformations)

Given the rectangular coordinate vector

$$\boldsymbol{A} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{a}_x - \frac{yz}{\sqrt{x^2+y^2+z^2}}\,\boldsymbol{a}_z$$

(a.) transform the vector $\boldsymbol{A}$ into cylindrical and spherical coordinates.
(b.) transform the rectangular coordinate point P (1,3,5) into cylindrical and spherical coordinates.
(c.) evaluate the vector $\boldsymbol{A}$ at P in rectangular, cylindrical and spherical coordinates.

(a.)

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

$$A_x = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{\rho}{\sqrt{\rho^2+z^2}}$$

$$A_z = -\frac{yz}{\sqrt{x^2+y^2+z^2}} = -\frac{z\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\rho}{\sqrt{\rho^2+z^2}} \\ 0 \\ -\dfrac{z\rho\sin\phi}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$A_\rho = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}} \qquad A_\phi = -\frac{\rho\sin\phi}{\sqrt{\rho^2+z^2}} = \qquad A_z = -\frac{z\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

$$\boxed{\boldsymbol{A} = \frac{\rho}{\sqrt{\rho^2+z^2}}\left(\cos\phi\,\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi - z\sin\phi\,\boldsymbol{a}_z\right)}$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$A_x = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$A_z = -\frac{yz}{\sqrt{x^2+y^2+z^2}} = -\frac{r^2\sin\theta\cos\theta\sin\phi}{r}$$
$$= -r\sin\theta\cos\theta\sin\phi$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \\ -r\sin\theta\cos\theta\sin\phi \end{bmatrix}$$

$$A_r = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi$$
$$A_\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi$$
$$A_\phi = -\sin\theta\sin\phi$$

$$\boxed{\begin{aligned}\boldsymbol{A} = {} & \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r \\ & + \sin\theta\cos\theta\left(\cos\phi + r\sin\theta\sin\phi\right)\boldsymbol{a}_\theta \\ & - \sin\theta\sin\phi\,\boldsymbol{a}_\phi\end{aligned}}$$

(b.) $P\,(1, 3, 5) \quad \Rightarrow \quad x = 1, y = 3, z = 5$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + 3^2} = \sqrt{10} = 3.16$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$

$$z = z = 5$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35} = 5.92$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{1^2 + 3^2}}{5}\right) = 32.3^o$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$

$$\boxed{P\,(1, 3, 5) \quad \Rightarrow \quad P\,(3.16, 71.6^o, 5) \quad \Rightarrow \quad P\,(5.92, 32.3^o, 71.6^o)}$$

(c.)

$$\boldsymbol{A}(1,3,5) = \frac{\sqrt{1^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}}\,\boldsymbol{a}_x - \frac{(3)(5)}{\sqrt{1^2 + 3^2 + 5^2}}\,\boldsymbol{a}_z = \boxed{0.535\,\boldsymbol{a}_x - 2.54\,\boldsymbol{a}_z}$$

$$\boldsymbol{A}(3.16, 71.6^o, 5) = \frac{3.16}{\sqrt{3.16^2 + 5^2}}\,(\cos 71.6^o\,\boldsymbol{a}_\rho - \sin 71.6^o\,\boldsymbol{a}_\phi$$

$$- 5 \sin 71.6^o\,\boldsymbol{a}_z\,)$$

$$= \boxed{0.169\,\boldsymbol{a}_\rho - 0.507\,\boldsymbol{a}_\phi - 2.53\,\boldsymbol{a}_z}$$

$$\boldsymbol{A}(5.92, 32.3^o, 71.6^o) =$$

$$\sin 32.3^o\,(\,\sin 32.3^o \cos 71.6^o - 5.92 \cos^2 32.3^o \sin 71.6^o\,)\,\boldsymbol{a}_r$$

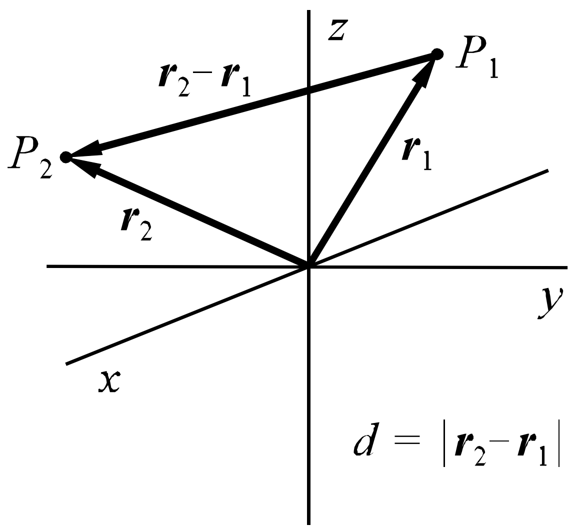
$$+ \sin 32.3^o \cos 32.3^o\,(\,\cos 71.6^o + 5.92 \sin 32.3^o \sin 71.6^o\,)\,\boldsymbol{a}_\theta$$

$$- \sin 32.3^o \sin 71.6^o\,\boldsymbol{a}_\phi$$

$$= \boxed{-2.05\,\boldsymbol{a}_r + 1.50\,\boldsymbol{a}_\theta + 0.507\,\boldsymbol{a}_\phi}$$

Separation Distances

Given a vector $\boldsymbol{r}_1$ locating the point P_1 and a vector $\boldsymbol{r}_2$ locating the point P_2, the distance d between the points is found by determining the magnitude of the vector pointing from P_1 to P_2, or vice versa.



$$d = |\boldsymbol{r}_2 - \boldsymbol{r}_1| = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$$

<u>Rectangular</u>

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

<u>Cylindrical</u>

$$d = \sqrt{\rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

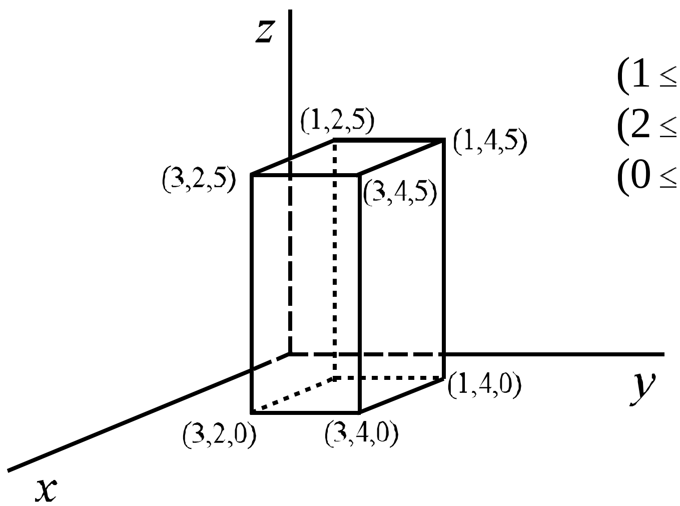
<u>Spherical</u>

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\phi_2 - \phi_1)}$$

Volumes, Surfaces and Lines in Rectangular, Cylindrical and Spherical Coordinates

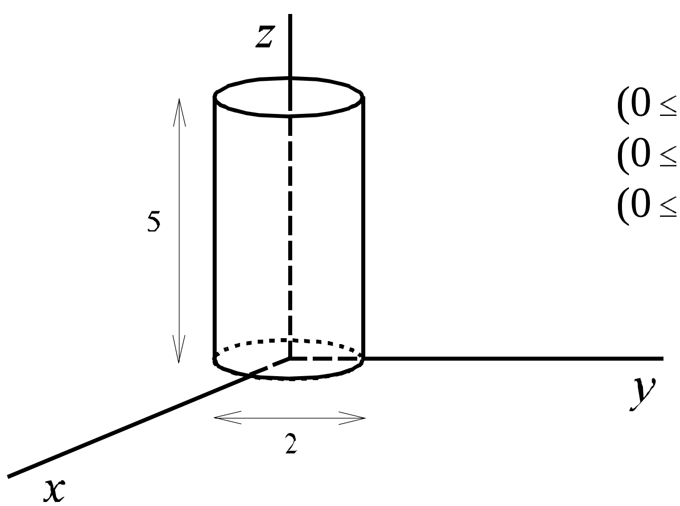
We may define particular three-dimensional volumes in rectangular, cylindrical and spherical coordinates by specifying ranges for each of the three coordinate variables.

<u>Rectangular volume</u> (2×2×5 box)



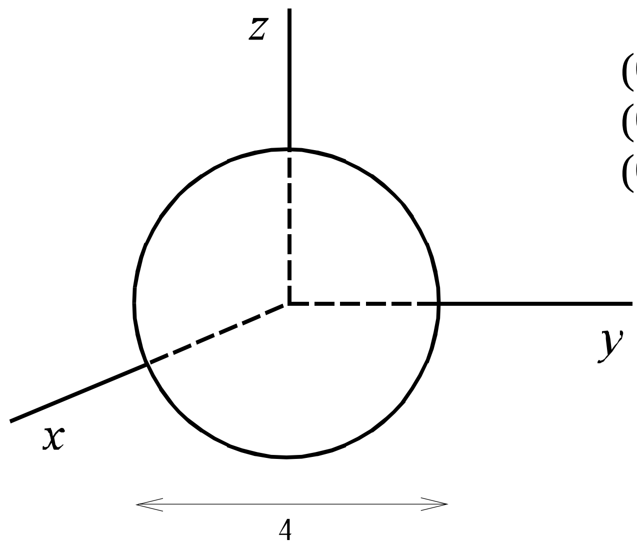
$(1 \leq x \leq 3)$
$(2 \leq y \leq 4)$
$(0 \leq z \leq 5)$

<u>Cylindrical volume</u> (cylinder of length = 5, diameter = 2)



$(0 \leq \rho \leq 1)$
$(0 \leq \phi \leq 2\pi)$
$(0 \leq z \leq 5)$

Spherical volume (sphere of diameter = 4)



$(0 \le r \le 2)$
$(0 \le \theta \le \pi)$
$(0 \le \phi \le 2\pi)$

Specific lines and surfaces can be generated in a given coordinate system according to which coordinate variable(s) is(are) held constant. A surface results when one of the coordinate variables is held constant while a line results when two of the coordinate variables are held constant.

Surface on the Rectangular volume (front face of the box)

$x = 3$ $\qquad$ $(x - \text{constant})$
$(2 \le y \le 4)$
$(0 \le z \le 5)$

Surface on the Cylindrical volume (upper surface of the cylinder)

$(0 \le \rho \le 1)$
$(0 \le \phi \le 2\pi)$
$z = 5$ $\qquad$ $(z - \text{constant})$

<u>Surface on the Spherical volume</u> (outer surface of the sphere)

$r = 2$ ($r -$ constant)
$(0 \le \theta \le \pi)$
$(0 \le \phi \le 2\pi)$

<u>Line on the Rectangular volume</u> (upper edge of the front face)

$x = 3$ ($x -$ constant)
$(2 \le y \le 4)$
$z = 5$ ($z -$ constant)

<u>Line on the Cylindrical volume</u> (outer edge of the upper surface)

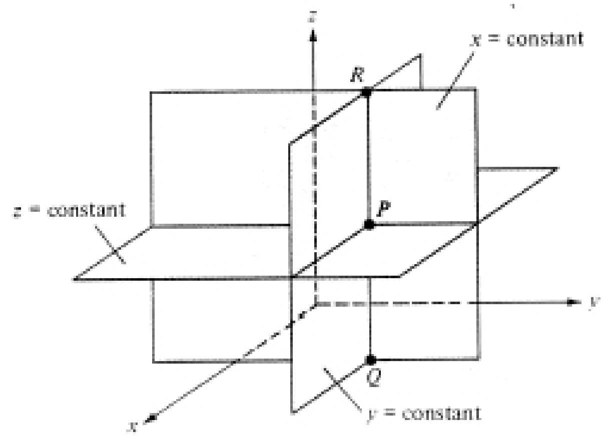
$\rho = 1$ ($\rho -$ constant)
$(0 \le \phi \le 2\pi)$
$z = 5$ ($z -$ constant)

<u>Line on the Spherical volume</u> (equator of the sphere)

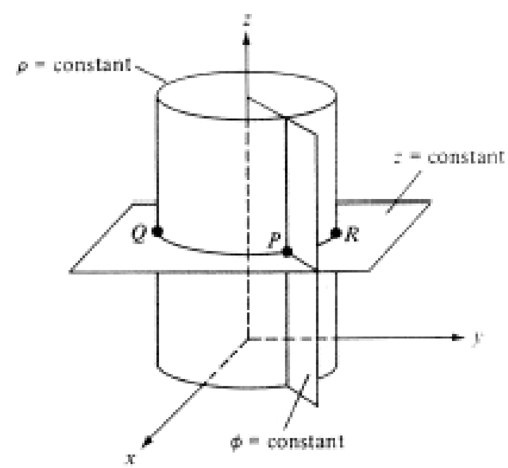
$r = 2$ ($r -$ constant)
$\theta = \pi/2$ ($\theta -$ constant)
$(0 \le \phi \le 2\pi)$

Constant Coordinate Surfaces

Rectangular Coordinates



Cylindrical Coordinates



Spherical Coordinates